A Discrete Singular Convolution Method for the Seepage Analysis in Porous Media with Irregular Geometry

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Abstract

A novel discrete singular convolution (DSC) formulation is presented for the seepage analysis in irregular geometric porous media. The DSC is a new promising numerical approach which has been recently applied to solve several engineering problems. For a medium with regular geometry, realizing of the DSC for the seepage analysis is straight forward. But DSC implementation for a medium with irregular geometry encounters some challenging issues. To overcome the difficulty, a novel DSC scheme for seepage analysis in irregular geometric porous media is proposed. There is no general analytical solution for the seepage analysis in irregular geometries; thus, the validation of the proposed algorithm is carried out by comparing the results with those from available numerical methods. Good agreement between the results shows that the proposed algorithm can be utilized in solving seepage analysis as a new approach.

Keywords: Discrete Singular Convolution, Seepage, Numerical Analysis, Irregular Geometric Medium

1. Introduction

One of the important problems in water resource management discipline is the seepage analysis in porous media. Water leakage in the body of the soil dams, seepage from the body of the channel and foundation of the concrete dams and water movement in aquifers are some of the practical examples in this branch. Apart from a few cases with special boundary conditions, governing equations on water seepage have no analytical solutions. Regarding this matter, numerical approaches can be utilized as proper instruments for solving the equations. Recently, one of the novel approaches which is used to solve the problems is discrete singular convolution (DSC). This method was invented to solve the differential equations by Wei (1999). After then, this method was implemented to solve a wide variety of engineering problems. Firstly, Wei enlisted the DSC to solve vibration problems in the field of solid mechanics (2001c). Then Civalek developed the method in 3D problems (2007). Navier-Stokes equations were analyzed using the DSC by Wan and Wei (2000). Xin et-al. analyzed nonlinear circuits in the field of electromagnetic (2004). Solving the quantum equations (Wei 2000) and edge detecting (Hou and Wei 2002), are other applications of the DSC. More recently, the authors of this paper developed the DSC algorithm for non-Darcian seepage analysis in the coarse porous media (Zayeri Baghlani Nejad and Shokrollahi 2011). A wide number of reviews about the DSC works can be found in Reference Shokrollahiand Zayeri Baghlani Nejad (2014).

DSC formulation which is presented in reference (Zayeri Baghlani Nejad and Shokrollahi 2011) is limited to media with rectangular geometries. However, in a great number of real problems, such as ground water movement, water seepage from the body of the soil dams and similar problems, the solution domain doesn’t have rectangular geometry. The objective of this paper is developing the DSC formulation for seepage analysis in the media with arbitrary irregular geometries. To this end, the method which was enlisted by Charles and Moinuddin for developing differential quadrature method in analyzing vibration of the non-rectangular plates is applied (1996).

In continue details of the DSC method will be discussed firstly. Then, formulation of the mentioned approach will be presented in the irregular domains. After that, discretization and solution of the governing equations of the water movement in porous media will be implemented. A computer program will be written for modeling some problems. Finally, the DSC results will be compared with those of the finite element method to validate the algorithm.

2. DSC method

Discrete singular convolution (DSC) method is a relatively new numerical technique in applied mechanics which was originally introduced by Wei (1999). Since then, the DSC method applied to various science and engineering problems. Accurate results and exact convergence have demonstrated that the DSC is a reliable and convenient numerical approach. The mathematical foundation of the DSC algorithm is the theory of distributions and wavelet analysis. Like some other numerical methods, the DSC method discretizes the spatial derivatives and, therefore, reduces the given partial differential equations into a system of linear algebraic equations. So, in the DSC algorithm, any function $f(x)$ and its derivatives with respect to a coordinate at a grid point x are approximated by a linear sum of the functional values in the narrow domain [x-x_{-M}, x+x_M] in that coordinate direction. This expression can be written as follows (Wei 1999):

$$f^{(n)}(x) \approx \sum_{i=-M}^{M} \delta_{\Delta,\sigma}^{(n)}(x-x_i)\, f(x_i) \tag{1}$$

where superscript n (n = 0, 1, 2...) denotes the nth-order derivative with respect to x. The 2M +1 is the computational bandwidth which is usually smaller than the whole computational domain. Therefore, the resulting approximation matrix has a banded structure, which makes the DSC method more efficient than normal global methods and is particularly valuable with respect to large scale computations. $\{x_i\}$ is an appropriate set of discrete points on which the DSC of Eq. (1) is well-defined and δ is a singular kernel. The DSC algorithm can be realized by using many approximation kernels (Wei et al. 2002b). However, it was shown (Wei 2001b; Wei et al. 2002a; Wei 2000; Wei 2001a) that for many problems, the use of the Regularized Shannon Kernel (RSK) is very efficient. The RSK is given by (Wei 1999):

$$\delta_{\Delta,\sigma}(x-x_k) = \frac{\sin\left(\frac{\pi}{\Delta}(x-x_k)\right)}{\frac{\pi}{\Delta}(x-x_k)} \exp\left(-\frac{(x-x_k)^2}{2\sigma^2}\right) \tag{2}$$

In these equations, $\Delta = L/(N-1)$ is the grid spacing and N is the number of grid points. The parameter σ determines the width of the Gaussian envelope and often varies in association with the grid spacing, i.e., $\sigma = r.\Delta$, here r is a parameter chosen in computations.

First and second derivative of RSK with respect to x is defined as below:

$$\delta_{\Delta,\sigma}^{(1)}(x_m-x_k) = \frac{\cos\frac{\pi}{\Delta}(x-x_k)}{(x-x_k)} \exp\left(\frac{-(x-x_k)^2}{2\sigma^2}\right) - \frac{\sin\frac{\pi}{\Delta}(x-x_k)}{\frac{\pi}{\Delta}(x-x_k)^2} \exp\left(\frac{-(x-x_k)^2}{2\sigma^2}\right) \tag{3}$$

$$\delta_{\Delta,\sigma}^{(2)}(x_m - x_k) = -\frac{(\frac{\pi}{\Delta})\sin\frac{\pi}{\Delta}(x-x_k)}{(x-x_k)}\exp\left(\frac{-(x-x_k)^2}{2\sigma^2}\right) - 2\frac{\cos\frac{\pi}{\Delta}(x-x_k)}{(x-x_k)^2}\exp\left(\frac{-(x-x_k)^2}{2\sigma^2}\right)$$
$$-2\frac{\cos(\frac{\pi}{\Delta})(x-x_k)}{\sigma^2}\exp\left(\frac{-(x-x_k)^2}{2\sigma^2}\right) + 2\frac{\sin\frac{\pi}{\Delta}(x-x_k)}{\frac{\pi}{\Delta}(x-x_k)^3}\exp\left(\frac{-(x-x_k)^2}{2\sigma^2}\right) \tag{4}$$
$$+\frac{\sin\frac{\pi}{\Delta}(x-x_k)}{\frac{\pi}{\Delta}(x-x_k)\sigma^2}\exp\left(\frac{-(x-x_k)^2}{2\sigma^2}\right) + \frac{\sin\frac{\pi}{\Delta}(x-x_k)}{\frac{\pi}{\Delta}\sigma^4}(x-x_k)\exp\left(\frac{-(x-x_k)^2}{2\sigma^2}\right)$$

$$\delta_{\Delta,\sigma}^{(1)}(0) = 0 \tag{5}$$

$$\delta_{\Delta,\sigma}^{(2)}(0) = -\frac{1}{\sigma^2} - \frac{\pi^2}{3\Delta^2} \tag{6}$$

As the DSC kernel is symmetric, the DSC computation requires a total of M fictitious grid points (FPs) outside each edge. Furthermore, the solution carries out for the grids inside the domain, so FPs must be eliminated. More precisely, it requires function values on these FPs which could be determined from those inside the domain by applying the boundary condition equations. Some attempts have been carried out for applying boundary conditions by researchers. Wei et al. (2001), Wei et al. (2002b), Zhao and Wei (2002), proposed a practical method to incorporate the boundary conditions. After that, Zhao et al. (2005) applied the iteratively matched boundary method to impose the free boundary conditions for Euler beams. More recently, Wang and Xu (2010) present a method for applying boundary conditions using the Taylor’s series expansion. For gaining more details about the DSC method, interested readers may refer to the works of Wei et al. (2002a), Wei et al. (2002b); Wei (2001c), Xiang et al. (2002), Wei (2000), Wei (2001a) and Civalek (2008).

3. Formulation of the DSC in irregular geometric media

If the calculation in a domain with irregular geometry is to be considered in a global cartesian coordinate (Fig. 1-a), with a mapping this in a natural coordinate system ξ and η (Fig. 1-b), the calculation will be simplified. Relations between the points in two domains define as follows (Charles and Moinuddin 1996):

$$x = \sum_{k=1}^{N_s} S_i(\xi,\eta)x_i \qquad y = \sum_{k=1}^{N_s} S_i(\xi,\eta)y_i \tag{7}$$

where x_i and y_i $(i = 1,2,...,N_s)$ are the coordinates of boundary nodes and $S_i(\xi,\eta)$ are interpolation functions. The mapping of an irregular medium is shown in Fig. 1 by use of 12 node transformation.

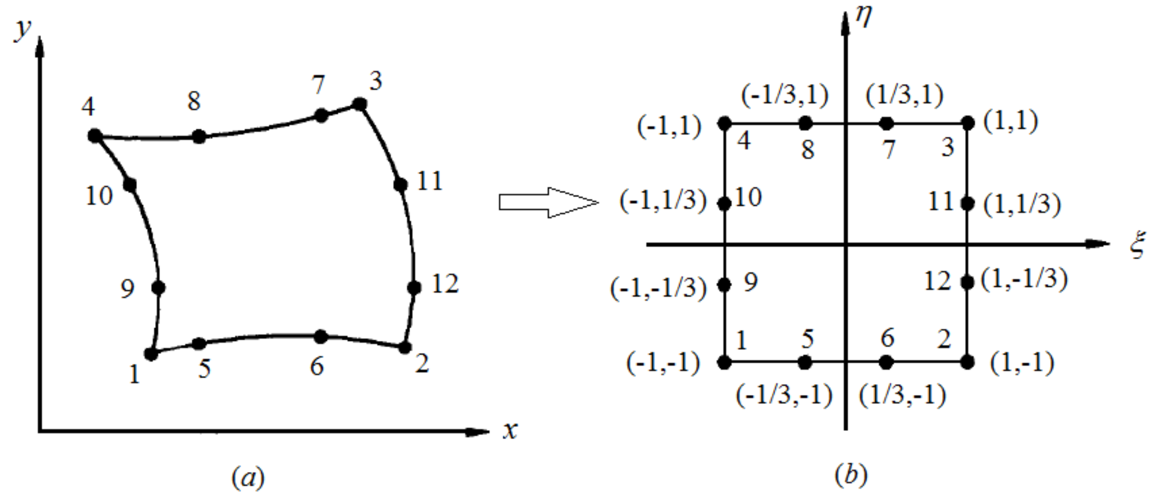

Fig. 1 12 node transformation mapping of a medium with irregular geometry

The S_i function must be defined in a way that its value will be considered as unit at the ith node and zero in the other $(N_s - 1)$ nodes. For example, if we use eight nodes for mapping of the geometry of the medium, it is convenient to use of the following functions (Han and Liew 1997):

$$
\begin{aligned}
S_i(\xi,\eta) &= \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1) && i=1,3,5,7; \\
S_i(\xi,\eta) &= \frac{1}{2}(1-\xi^2)(1+\eta\eta_i) && i=2,6; \\
S_i(\xi,\eta) &= \frac{1}{2}(1+\xi\xi_i)(1-\eta^2) && i=4,8
\end{aligned} \tag{8}
$$

where ξ_i and η_i are the coordinates of the ith node in the $\xi-\eta$ coordinate system.

Based on the chain rule of the derivative, the relation between first and second derivatives in the two coordinate systems can be written as follows (Han and Liew 1997):

$$
\begin{aligned}
&\begin{Bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{Bmatrix} = J_0^{-1} \begin{Bmatrix} \partial f/\partial \xi \\ \partial f/\partial \eta \end{Bmatrix} \\
&\begin{Bmatrix} \partial^2 f/\partial x^2 \\ \partial^2 f/\partial y^2 \\ 2\partial^2 f/\partial x\partial y \end{Bmatrix} = J_2^{-1} \begin{Bmatrix} \partial^2 f/\partial \xi^2 \\ \partial^2 f/\partial \eta^2 \\ \partial^2 f/\partial \xi\partial \eta \end{Bmatrix} - J_2^{-1} J_1 J_0^{-1} \begin{Bmatrix} \partial f/\partial \xi \\ \partial f/\partial \eta \end{Bmatrix}
\end{aligned} \tag{9}
$$

where J_0 , J_1 and J_2 matrixes are:

$$J_0 = \begin{bmatrix} \partial x/\partial\xi & \partial y/\partial\xi \\ \partial x/\partial\eta & \partial y/\partial\eta \end{bmatrix} \qquad J_1 = \begin{bmatrix} \partial^2 x/\partial\xi^2 & \partial^2 y/\partial\xi^2 \\ \partial^2 x/\partial\eta^2 & \partial^2 y/\partial\eta^2 \\ \partial^2 x/\partial\xi\partial\eta & \partial^2 y/\partial\xi\partial\eta \end{bmatrix}$$

$$J_2 = \begin{bmatrix} (\partial x/\partial\xi)^2 & (\partial y/\partial\xi)^2 & (\partial x/\partial\xi)(\partial y/\partial\xi) \\ (\partial x/\partial\eta)^2 & (\partial y/\partial\eta)^2 & (\partial x/\partial\eta)(\partial y/\partial\eta) \\ (\partial x/\partial\xi)(\partial x/\partial\eta) & (\partial y/\partial\xi)(\partial y/\partial\eta) & \frac{1}{2}((\partial x/\partial\xi)(\partial y/\partial\eta)+(\partial x/\partial\eta)(\partial y/\partial\xi)) \end{bmatrix} \tag{10}$$

The following matrixes can be defined for simplification:

$$A = J_0^{-1} = [a_{ij}]_{2\times 2} \qquad B = J_2^{-1} = [b_{ij}]_{3\times 3} \qquad T = J_2^{-1}J_1J_0^{-1} = [t_{ij}]_{3\times 2} \tag{11}$$

Approximation of the *n*th-order derivative of the *f* function (equation 4) can be mapped from $x-y$ coordinate system to the $\xi-\eta$ coordinate system, using equations (7) to (10). For instance, the first order derivative of the f function using new DSC formulation in the $\xi-\eta$ coordinate system is as follows:

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_i} = a_{11}\sum_{k=-m}^{m}\delta_{\Delta,\sigma}^{(1)}(k\Delta\xi)\,f(\xi_k) + a_{12}\sum_{k=-m}^{m}\delta_{\Delta,\sigma}^{(1)}(k\Delta\eta)\,f(\eta_k) \tag{12}$$

4. Governing equations

The governing equation of water movement in porous media is the Richards equation (Richards 1931):

$$\frac{\partial(\rho_w V_x)}{\partial x} + \frac{\partial(\rho_w V_y)}{\partial y} + \frac{\partial(\rho_w V_z)}{\partial z} + \frac{\partial(\rho_w n S_w)}{\partial t} = 0 \tag{13}$$

In the above equation V is the current velocity, ρ_w is the mass density of water, n is the void ratio of soil, S_w is the saturation of soil and t is the time parameter.

Despite the fact that all the physical systems are 3D, as the water movement in parallelogram vertical planes are similar, the Z coordinate is eliminated from the calculation for simplicity (Harr 1962). So, assuming the saturated medium and steady stream, the continuity equation will be as follows:

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0 \tag{14}$$

The combination of Darcy's law and continuity equation will due to the following equation for modeling of seepage in homogenous porous media.

$$k_x\frac{\partial^2 h}{\partial x^2} + k_y\frac{\partial^2 h}{\partial y^2} = 0 \tag{15}$$

where h is piezometric head and k_x and k_y are hydraulic conductivities in x and y directions, respectively.

For solving the seepage analysis in a porous medium, the solution domain must be mapped to a regular medium and then it must be meshed. After that, N_ξ and N_η are the number of equidistant nodes in both directions of new coordinate system. The distance between the nodes are Δ_ξ and Δ_η. The governing equation (15) will be descretized at these nodes.

The discrete form of the governing equation (15) in the natural coordinate system for every arbitrary node (ξ_i,η_j) could be shown as follows:

$$\begin{aligned}&(k_x b_{11}+k_y b_{21})_{ij}\sum_{k=-m}^{m}\delta_{\Delta,\sigma}^{(2)}(k\Delta\xi)h_{i+k,j}+(k_x b_{12}+k_y b_{22})_{ij}\sum_{k=-m}^{m}\delta_{\Delta,\sigma}^{(2)}(k\Delta\eta)h_{i,j+k}+\\&(k_x b_{13}+k_y b_{23})_{ij}\sum_{k=-m}^{m}\delta_{\Delta,\sigma}^{(1)}(k\Delta\xi)\times\sum_{k=-m}^{m}\delta_{\Delta,\sigma}^{(1)}(k\Delta\eta)h_{i,j+k}-(k_x t_{11}+k_y t_{21})_{ij}\times\\&\sum_{k=-m}^{m}\delta_{\Delta,\sigma}^{(1)}(k\Delta\xi)h_{i+k,j}-(k_x t_{12}+k_y t_{22})_{ij}\sum_{k=-m}^{m}\delta_{\Delta,\sigma}^{(2)}(k\Delta\eta)h_{i,j+k}=0\end{aligned} \qquad (16)$$

The equations of the boundary conditions depend on the problem. Boundary conditions could be carried out after writing these equations in the new coordinate system and descretizing them. For example, the boundary condition at a boundary parallel with y direction is:

$$\frac{\partial h}{\partial x}=0 \qquad (17)$$

The discrete form of the equation (17) in the transformed coordinate system is:

$$a_{11}\sum_{k=-m}^{m}\delta_{\Delta,\sigma}^{(1)}(k\Delta\xi)h_{i+k,j}+a_{12}\sum_{k=-m}^{m}\delta_{\Delta,\sigma}^{(1)}(k\Delta\eta)h_{i,j+k}=0 \qquad (18)$$

The boundary condition could be easily participated in the calculation by use of the equation (18).

Implementing the equation (16) for every node over the solution domain and applying the boundary conditions due to construction an equation system that its matrix form is as follows:

$$[C]\{h\}=\{d\} \qquad (19)$$

where $\{h\}=\{h_{1,1},h_{1,2},\ldots\ldots,h_{N\xi\times N\eta}\}$. Solving equations (19) produces water head at each node.

5. Results

To confirm the analysis and validate the proposed algorithm, an example is solved using the modified DSC. The obtained results are compared with those of the conventionally finite element method. Details of the problem and procedure of solution are illustrated below:

Example: plan of a farm which has a pool for fish growing on its center is shown in Fig. 2. The water head of the pool with respect to the base level is considered as 7 m. There are two irrigation channels in north and south sides which their water head is 10 m with respect to the base level. The objective is to calculate the water head at points of the field. In this example we consider that $k_x=k_y$, so paying attention to the equation (16), these coefficients will be eliminated from the solution.

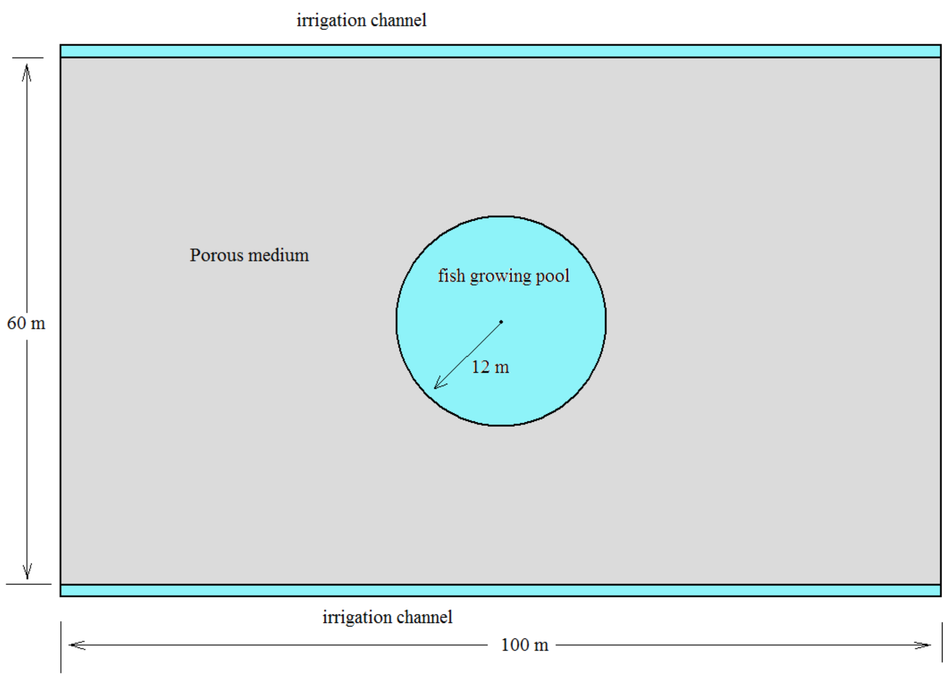

Fig. 2 Geometric specifications of the example

As the shape and the boundary conditions of the problem are symmetric, we can carry out the calculations for a quarter of the solution domain (Fig. 3). For solving the problem using the modified DSC, 8-node transform is employed. Fig. 3 shows the boundary conditions and boundary nodes for mapping the solution domain.

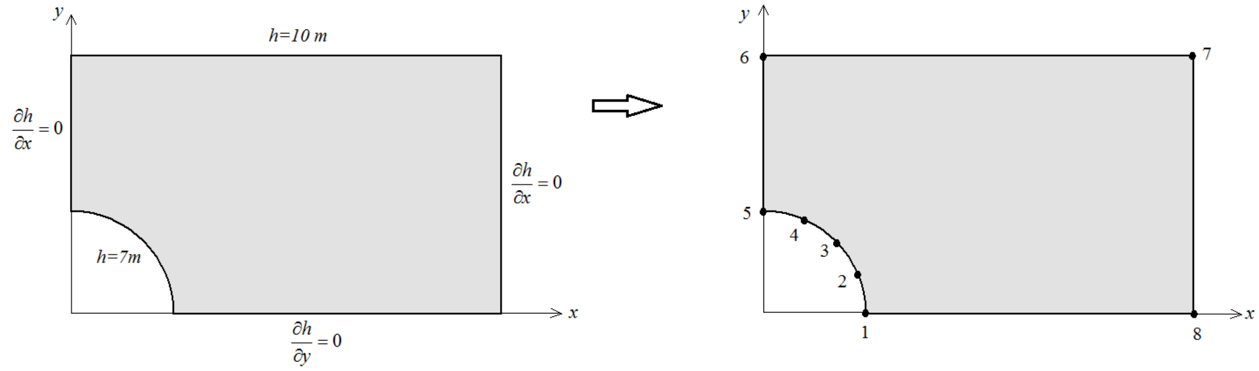

Fig. 3 Boundary conditions and locations of the transformation nodes for mapping of a quarter of the solution domain

Using the written computer program, *h* values are calculated at various nodes of the domain. Fig. 4 shows the equipotential lines obtained from the new numerical model. Comparison between the results with those of finit element method is utilized for validation. For achieving to this end, the water head obtained from two methods is exhibited for some chosen point in table (1). The coordinates of points in this table are those that devoted to the Fig. 4. As it is seen, there is a good agreement between the results of proposed algorithm and finite element method. This matter shows the accuracy of the new DSC algorithm.

Table1 Comparison of the water heads obtained from the DSC and finite element methods at some nodes in (x, y) coordinate system.

ERROR	h_{DSC} (m)	h_{FEM} (m)	Y (m)	X (m)
3.13E-05	9.5947	9.5944	8.02	16.04
1.06E-05	9.4092	9.4093	12.04	16.10
1.08E-05	9.2441	9.2440	16.09	16.23
2.20E-05	9.1070	9.1068	20.15	16.46
0.00E+00	9.0041	9.0041	24.24	16.80
2.10E-05	9.5332	9.5330	8.00	20.01
3.41E-05	8.8053	8.8050	19.08	24.49
1.16E-05	8.6460	8.6459	22.31	24.98
1.19E-05	8.4187	8.4186	24.67	27.15
2.13E-05	9.3728	9.3726	7.84	27.95
0.00E+00	9.0832	9.0832	11.65	27.99
0.00E+00	8.8161	8.8161	15.38	28.09
5.90E-05	8.4705	8.4700	21.33	28.14
1.17E-05	8.5796	8.5795	18.73	28.61
2.46E-05	8.1354	8.1352	23.77	30.72
0.00E+00	8.7230	8.7230	11.99	35.48
2.19E-05	9.1257	9.1255	8.05	36.01
0.00E+00	8.1892	8.1892	16.22	36.59
1.30E-05	7.7195	7.7194	20.12	37.19
3.61E-05	8.3104	8.3101	13.65	39.60
1.13E-05	8.8805	8.8804	9.01	40.16
3.95E-04	7.6020	7.5990	17.73	41.25

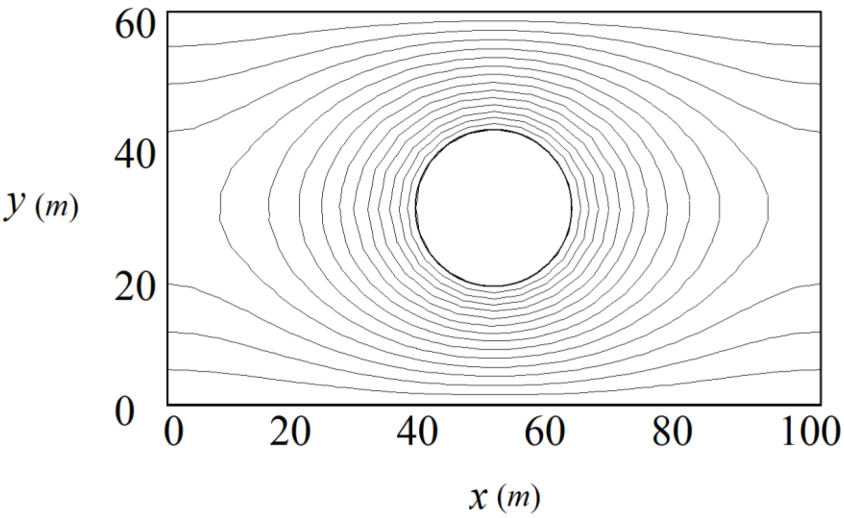

Fig. 4 Equipotential lines obtained from the DSC method in the entire of the solution domain

6. Conclusion

In this paper a novel formulation of the DSC is presented for seepage analysis in media with irregular geometry. The governing equations and boundary conditions are discretized using the proposed approach and a computer program is produced for solving some examples. A problem with irregular geometry solved by use of a new algorithm and the results are compared with those of finite element method. Good agreement between the results demonstrated that the proposed algorithm could be utilized as a promising approach to solve the seepage through porous media.

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