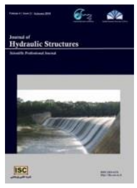
Journal of Hydraulic Structures
J. Hydraul. Struct., 2022; 8(2):27-39
DOI: 10.22055/jhs.2022.41574.1225



OPEN ACCESS

Fractures in non-homogeneous rockfill materials from a micromechanics perspective

Saber Alidadi [1]
Rasoul Alipour [2]
Mohammadreza Shakeri [1]

Abstract

This study investigates particle breakage and cracks propagation of non-homogeneous rockfill materials, particularly conglomerates, from a microscale perspective. The conglomerate's materials were gathered from Masjed-E-Soleyman, MES, rockfill dam, Iran. The study of particle breakage in rockfill material has been investigated by several researchers worldwide, both in the laboratory and through numerical simulations. However, the previous research focused on homogeneous rockfill materials, not non-homogeneous ones. The first part of this research investigates crack propagation in conglomerates due to high-stress conditions in an embankment dam. The second part of the paper evaluated the effects of crack propagation on the MES dam crest settlement. In this paper, sets of XRD tests were performed to investigate the microstructure of conglomerates rockfill, which revealed that calcite constitutes the majority of the rockfill structure. In accordance with geology science, the calcite has a high potential for breaking, and a numerical simulation was developed to illustrate the fractures and crack propagation in a high rockfill dam. The results of this research are useful for understanding the concept of large deformations that occurred in the MES dam and needed rehabilitation measures for preventing dam breakage.

Keywords: Crack propagation, Conglomerates, Particle breakage, Stress concentration, Rockfill dam.

Received: 10 August 2022; Accepted: 10 September 2022

1. Introduction

The post-construction settlement of rockfill dams and high-filled ground, a phenomenon of much importance, is mainly caused by the time-dependent particle breakage of the rockfill material [1]. This phenomenon has become a major threat to these infrastructures' serviceability,

[1] Department of Civil Engineering, Faculty of Civil and Earth Resources Engineering, Central Tehran Branch, Islamic Azad University, Tehran, Iran.
[2] Department of Civil Engineering, Faculty of Technical and Engineering, Shahrekord University, Shahrekord, Iran. Email: r.alipour@sku.ac.ir **(Correspond Author)**

maintenance, and safety. Rockfill materials' creep and particle breakage behavior have been widely studied in laboratory tests, constitutive modeling, Discrete Element Method DEM, and numerical simulation [2, 3].

A few studies have been performed on the propagation of cracks in non-homogeneous rockfills worldwide. Most studies have been conducted on homogeneous rockfill materials. Due to its complexity from a geometric and mechanics perspective, modeling crack propagation on non-homogeneous rockfill was difficult both in the laboratory and in numerical simulations.

The process of particle breakage affects the mechanical characteristics of rock fill and granular materials, which are extensively used in geotechnical engineering projects [1-3]. Particle breakage directly determines the behavior of rockfill structures such as influencing its dilatancy, friction angle, strength and permeability, generating creep, wetting and residual deformation under seismic load [7]. Many researchers have performed laboratory and numerical studies to investigate the relationship between pure homogenous rockfill particle breakage and constitutive characteristics [5-11]. The factors that affect the generation of rockfill particle breakage (such as moisture content, particle size, confining pressure, maximum deviatoric stress, maximum principal stress ratio, maximum dilatancy angle, peak friction angle and particle size (D_{10}, D_{15} and D_{50})) have been investigated [2, 8, 10,12-14], and the rules of rock- fill particle breakage under different loadings have also been investigated [15-19].

In this paper, the specifications of the MES rockfill dam were investigated in the first part. Then XRD analyses determine the chemical composition of the conglomerate materials. Finally, the crack propagation model with COMSOL software has been determined. It should be noted that this crack propagation in the conglomerate materials could produce large deformations and more problems in the operation of the MES rockfill dam.

2. Masjed-E-Soleyman rockfill dam

For this research, Masjed-E-Soleyman-Dam was selected as a case study. It is one of the highest rockfill dams in Iran at 177 meters in height. Additionally, the MES dam suffers from severe cracks and settlement in its crest, which Alidadi et al. [23] studied them. The rockfill used in the construction of the dam was a non-homogeneous rockfill known as conglomerates in the Bakhtiary formation. This research demonstrates that the conglomerate and its particle breakage could produce large part of this deformations. Conglomerates rockfill consists of a range of fine to coarse rocks that have been integrated with cementitious materials. Because of the complex structure of conglomerates, it is unclear how cracks propagate within the rockfill when particles break in different stress condition. The MES dam was constructed with conglomerates ranging in size from 150 mm to 1000 mm. Figure 1, illustrate the MES dam body and its spillway.

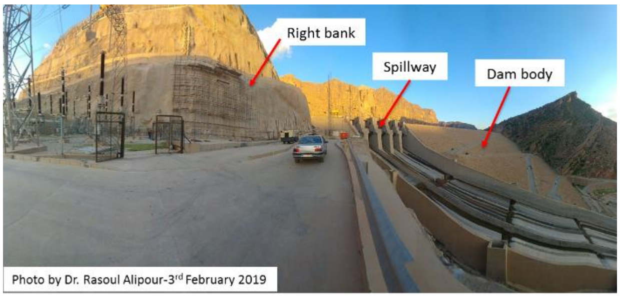


Fig. 1. Masjed-E-Soleyman rockfill dam, Iran

Figure 2 illustrate a general view of materials in the MES dam. Figure 2a, depicts large scale conglomerate which are used in the downstream of MES dam. Also figure 2b illustrates various types of conglomerate in the sieve analyses.

Fig. 2. Bakhtiary formation conglomerate, a) Rip-Rap of conglomerate at the downstream of MES dam, b) Separating crushed conglomerate in sieve analysis

3. XRD analyses

Calcite veins in the sedimentary rocks seal the natural fractures by crystal growth and play a very important role for rocks under compression. The interfaces between the veins and host rock are generally considered as structures determining the mechanical response and fracture of the rocks. Furthermore, thickness variation of the calcite vein also can affect the mechanical behavior and failure of the sedimentary rocks significantly.

According to Alidadi et al., 2022, the conglomerates of the MES dam are highly susceptible to fracture at maximum stress, with a breakage index of 42 percent. This research investigated the high fracture potential of conglomerates from a micromechanics perspective. Figures 3, 4, and 5 illustrate how the conglomerates structure contains a substantial amount of calcite, based on the results of the XRD laboratory analysis. X-Ray diffraction analysis (XRD) is a nondestructive technique that provides detailed information about the crystallographic structure, chemical composition, and physical properties of a material [24]. As mentioned previously, conglomerates rockfill is very brittle, so when the adjacent rock imposes a load on the contact surfaces, the conglomerate rock will break. There will be a rearrangement of whole particles as a result of the breakage, resulting in settlement and cracks in the crest dam. In this research, the main focus is on the propagation of cracks in conglomerates at three levels of the MES dam body. Tables 1, 2, and 3 illustrate XRD analyses of conglomerate in these samples.

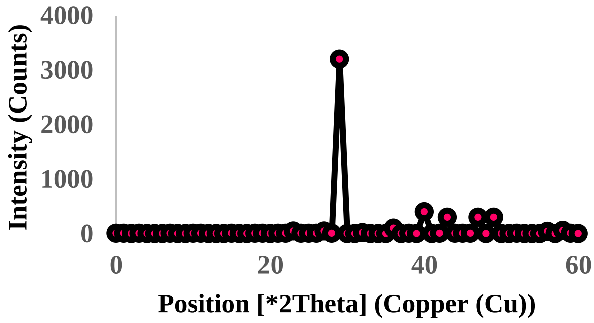


Fig. 3. XRD results for specimen 1 of conglomerates

Table 1. XRD analyses of conglomerate sample 1

Sample	Phases (s)
1 LAB: 1-10310	Calcite (05-0586) = 91%
Date 28.02.2022	Clay Mineral = 2%
kV = 40 mA = 30	Quartz (33-1161) = 6%
Ka = Cu	Dolomite (36-0426) = 0%

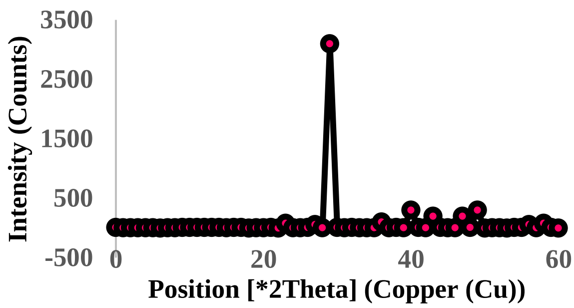


Fig. 4. XRD results for specimen 2 of conglomerates

Table 2. XRD analyses of conglomerate sample 2

Sample	Phases (s)
2 LAB: 2-10310	Calcite (05-0586) = 94%
Date 28.02.2022	Clay Mineral = 1%
kV = 40 mA = 30	Quartz (33-1161) = 3.5%
Ka = Cu Fe = Ni	Dolomite (36-0426) = 1%

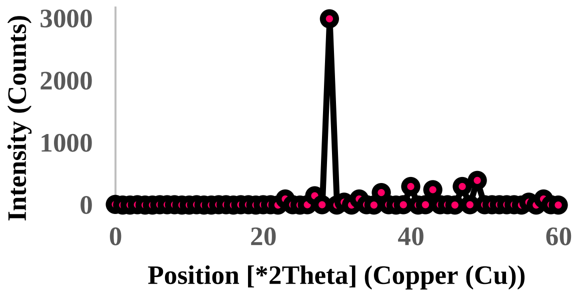


Fig. 5. XRD results for specimen 3 of conglomerates

Table 3. XRD analyses of conglomerate sample 3

Sample	**Phases (s)**
3 LAB: 3-10310	Calcite (05-0586) = 87%
Date 28.02.2022	Clay Mineral = 1.5%
kV = 40 mA = 30	Quartz (33-1161) = 6%
Ka = Cu Fe = Ni	Dolomite (36-0426) = 4%

According to figures 3, 4, and 5, since most of the conglomerate's structure consists of calcite, it has a high potential for fracture. Figure 6 illustrates the mathematical relationship between the potential breakage of conglomerates and the crest settlement of the MES dam [23]. As depicted in this picture, a conglomerate has the potential to breakage at 30%, which produces an 11.5m settlement.

Figure 6 depict that occurring crest settlement of MES rockfill dam just by particle breakage has potential to a settlement near an 11.5m. The current settlement of the crest is 6.5 m. This important result revealed that the dam deformation would continue and proper measures should take to prevent dam breakage.

Fig. 6. Prediction the relationship between the breakage index and crest settlement of the MES dam [23]

4. Mathematical concepts

The research was conducted with COMSOL Multiphysics software and MATLAB. COMSOL Multiphysics is a complete suite for modeling physical phenomena using the finite element method and MATLAB is well known for its powerful scripting capabilities. In the first step, COMSOL Multiphysics was used to create a finite element model of a particle of conglomerates with a diameter of 150 mm. As a second step, COMSOL Multiphysics was coupled with MATLAB for parametric sweeps of crack propagation. In order to improve the precision of the simulation, 400 edge meshes were selected in the induced crack length. An extremely fine mesh was also used to construct the particle size mesh. In view of the very high cracking potential of conglomerates, which was investigated in a laboratory test by Alidadi et al., 2022, the crack propagation analysis was conducted in the elastic phase [23]. According to Ambati et al., 2022, the phase field damage model is useful for modeling cracks in brittle geomechanic materials, such as concrete and conglomerates [25]. The behavior of the materials in numerical modelling has important effects in prediction the behavior of the materials [23, 24, 28].

4.1. Phase field damage model

Phase field damage models are closely related to strain-based damage models, but they are derived from a different starting point. This model is based on a regularization of Griffith's classical theory for brittle fracture, where a second-order phase field is used to regulate the geometric crack discontinuity, similar to the approach [29].With a viscous regularization of the crack phase field $\emptyset$, the energy functional governing crack propagation takes the following form:

$$E(x,t,u,\nabla u,\emptyset,\nabla\emptyset) = \int_{\Omega}^{\infty}(1-d(\emptyset))W_{s0}\,dx + \int_{\Omega}^{\infty}\eta\frac{\partial\emptyset}{\partial t}dx + \int_{\Omega}^{\infty}G_c\left(\frac{1}{2l_{int}}\emptyset^2 + \frac{l_{int}}{2}|\nabla\emptyset|^2\right)dx \quad (1)$$

In this case, $d(\emptyset)$ is the damage function, Ws_0 is the elastic strain energy density, η represents viscosity, G_c is the critical energy release rate, and lint represents the internal length scale that appears when discrete fractures are regularized.

Based on the two dependent variables, u and $\emptyset$, the main contributions to the coupled initial boundary value problem are as follows:

$$\begin{cases} \rho\frac{\partial^2 u}{\partial t^2} = \nabla,\sigma_2 + F_V & \text{on } \Omega \\ \tau\frac{\partial\emptyset}{\partial t} = \frac{\partial d(\emptyset)}{\partial t}H_d - \emptyset - l_{int}^2\nabla^2\emptyset & \text{on } \Omega \\ 0 = -\nabla\emptyset, n & \text{on } \Omega \end{cases} \quad (2)$$

In this case, the characteristic time τ and the state variable H_d have been introduced. Variable H_d should satisfy Kuhn-Tucker conditions. In mathematical optimization, the Karush–Kuhn–Tucker (KKT) conditions, also known as the Kuhn–Tucker conditions, are first derivative tests (sometimes called first-order necessary conditions) for a solution in nonlinear programming to be optimal. This optimization needs satisfaction of some regularity conditions. For this reason H_d should be a function of the crack driving force or D_d:

$$H_d(t,u) = \max_{\tau\in[0,t]} D_d(\tau,u) \quad (3)$$

The momentum balance in Equation 2 is included for completeness, but is not directly implemented in the phase field damage model. In spite of this, the equation implies that the damaged stress, as defined by Equation 3, is used to determine the mechanical equilibrium.

4.2. COMSOL-MATLAB Implementation

To load the specimen, Prescribed Displacement nodes are added to each Rigid Connector. As a result, the loading is displacement-controlled, which is essential for tracking peak loads in the global response. A load-controlled setup would have stopped the simulation before reaching the first peak.

The phase field damage model can often show poor or slow convergence when solving this type of brittle fracture problem using a fully coupled strategy. This example illustrates how a segregated strategy can be used to improve the convergence and stability of the numerical solution. The crack phase field and the displacement field are separated into two groups. For step n + 1, this type of algorithmic operator split was initially proposed in Miehe et al. [29].

1 - During the initialization process, the crack phase field, displacement field, and other state variables will be known.
2 - Update state variables, updating internal state variables with the values obtained at step n.
3 - Compute the crack phase field variable in Newton steps, freezing the displacement field at each step.
4 - Solve for the displacement field, compute the displacement field variable in a Newton step using the updated crack phase field.

By following these steps, a single-pass algorithm can be developed that is accurate only for sufficiently small changes in parameter values. This scheme may be improved by adding a multi-pass correction by iterating over steps 3 and 4 until convergence is achieved or until a predetermined number of iterations has been completed. A strategy of this type is used in this example by setting the Number of iterations to 3 in the Segregated subnode of the Stationary Solver.

It should be noted that the suggested solver configuration does not require convergence of the outer problem, and the solution is always accepted after three separate iterations. In spite of this, each subgroup meets the convergence criteria at the local level. Therefore, given the current crack phase field, the displacement field can be considered a converged solution.

It is possible to improve the accuracy of the scheme by increasing the number of iterations (or requiring the outer problem to be converged) if the extra computational cost is acceptable.

4.3. Model definition

Conglomerates particles are modeled as octagon, which are close to the original shape of the broken rockfill in the MES dam. A site investigation revealed that the majority of the conglomerates particles contain 70% cement and 30% cobbles. An example of an octagonal conglomerates particle is shown in Figure 7, with the cobbles being represented as circles. To propagate a crack, the first step is to induce a crack length that extends far from the particle's edge and forward to its center. An illustration of the induced crack length and the extremely fine mesh associated with the crack is shown in Figure 8. As illustrated in Table 4, conglomerates particles contain cement and cobbles that exhibit mechanical properties. In the phase field damage model, the critical release energy rate Gc is a critical parameter which has been calculated from the plastic work paradigm for the MES dam [23].

Table 4. The mechanical properties of conglomerates

Cementation part	
Young's modulus	7 [GPa]
Poisson's ration	0.3
Critical energy release rate	2200 J/m^2
Cobbles part	
Young's modulus	6 [GPa]
Poisson's ratio	0.2
Critical energy release rate	1800 J/m^2

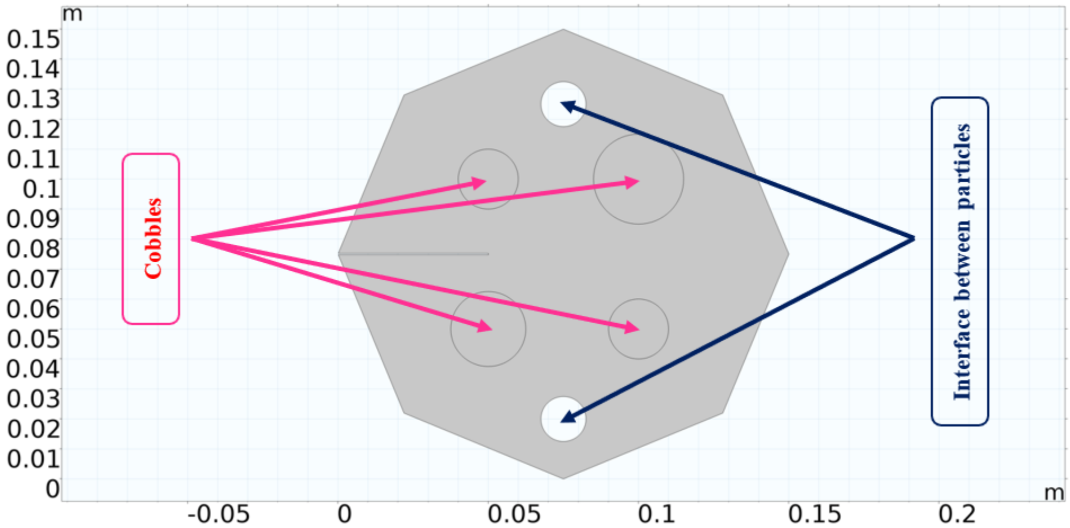


Fig. 7. An octagonal conglomerates particle's finite element geometry

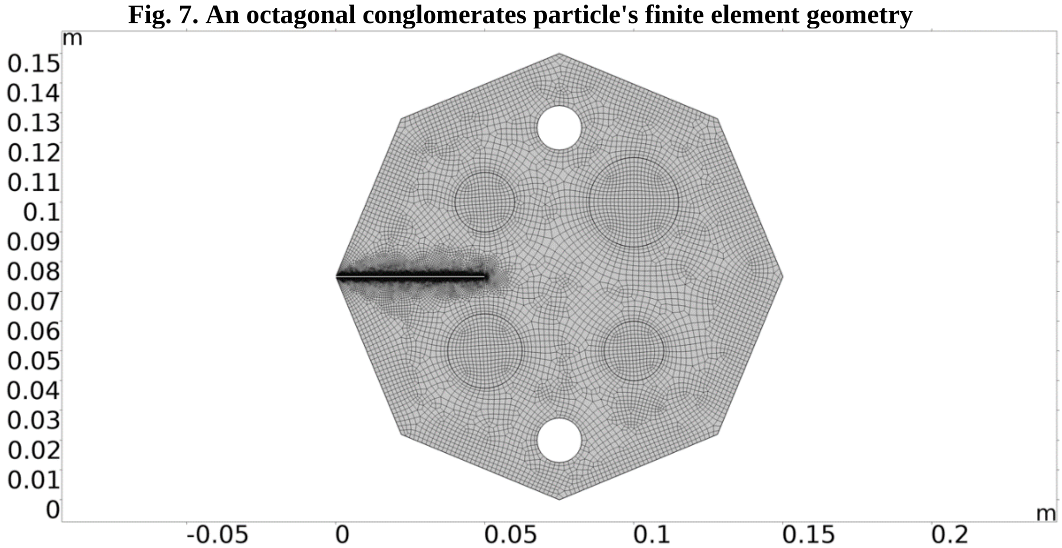


Fig. 8. An octagonal conglomerates particle's finite element mesh

The rockfill is modeled as an elastic material with damage in order to account for tensile cracking. In order to incorporate cracking into domain material, a phase field approximation is made after the sharp crack geometry has been determined. As the phase field damage model is not set up to include a damage threshold, all material points subjected to tension will be damaged. In the damage model, the only material input is the critical energy release rate Gc, whereas the tensile strength is determined by the damage evolution function and the phase field evolution. A localized phase field can be modified by the internal length scale l_{int}. A parameter of 0.05 mm has been set for l_{nt} this study.

To implement crack propagation, parametric sweep methods were used using MATLAB and its compatibility with COMSOL Multiphysics. We assume that the particles loaded on the crack edge are on three elevations of the dam, which illustrates the total stress on the crack edge. As a result of this stress on the crack edge, the crack developed and spread across the conglomerates particles.

5. Result and discussion

According to figure 9, three elevations have been chosen for calculating crack propagation. Figure 10 illustrates a stress-displacement diagram for three elevations of the MES dam with three peaks. Each peak represents the beginning of the crack propagation and the beginning of the crack tip displacement. Once the crack has begun propagating, as shown in Figure 10, the reaction stress at the crack tip gradually reduces toward zero as the crack progresses. A full crack propagation is illustrated in Figure 11 at the point of highest stress, which is the deepest point from the dam crest with a total stress of approximately 3.6 MPa. As discussed in Alidadi et al. [23], this study primarily focuses on the crack propagation in the conglomerates rockfill, which caused the particle breakage phenomenon that resulted in the dam settlement.

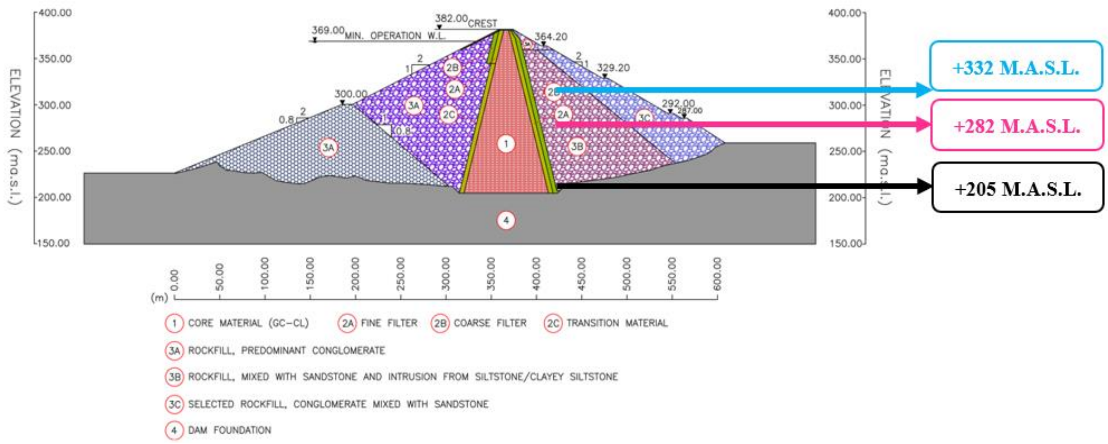


Fig. 9. The three elevations chosen for the calculation of crack propagation.

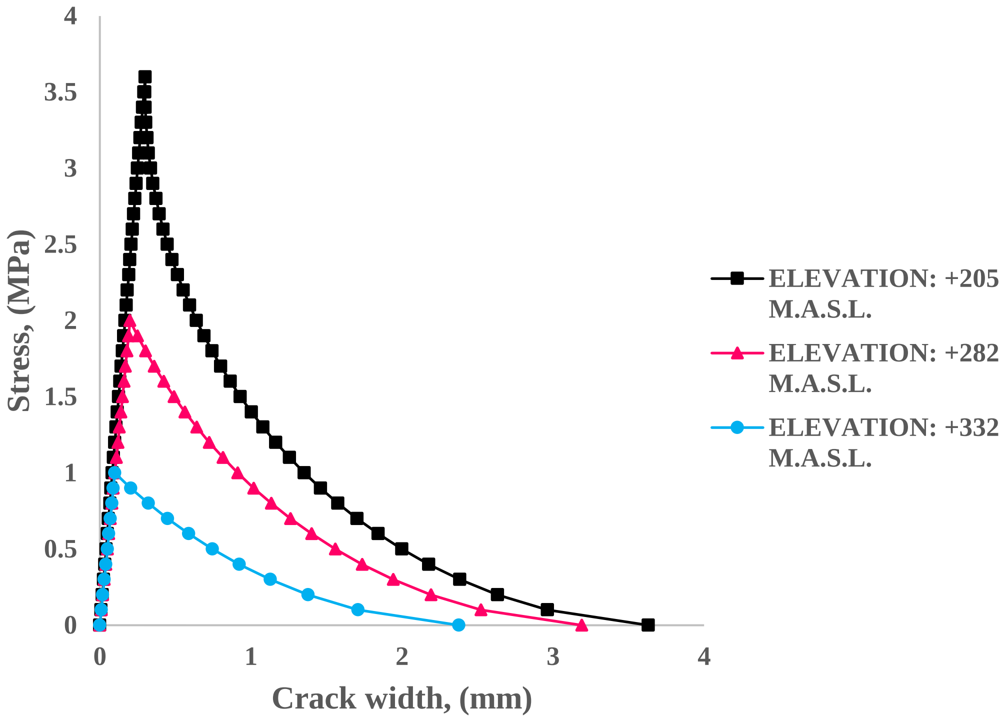


Fig. 10. A stress-displacement analysis of conglomerates at three elevations of the MES dam

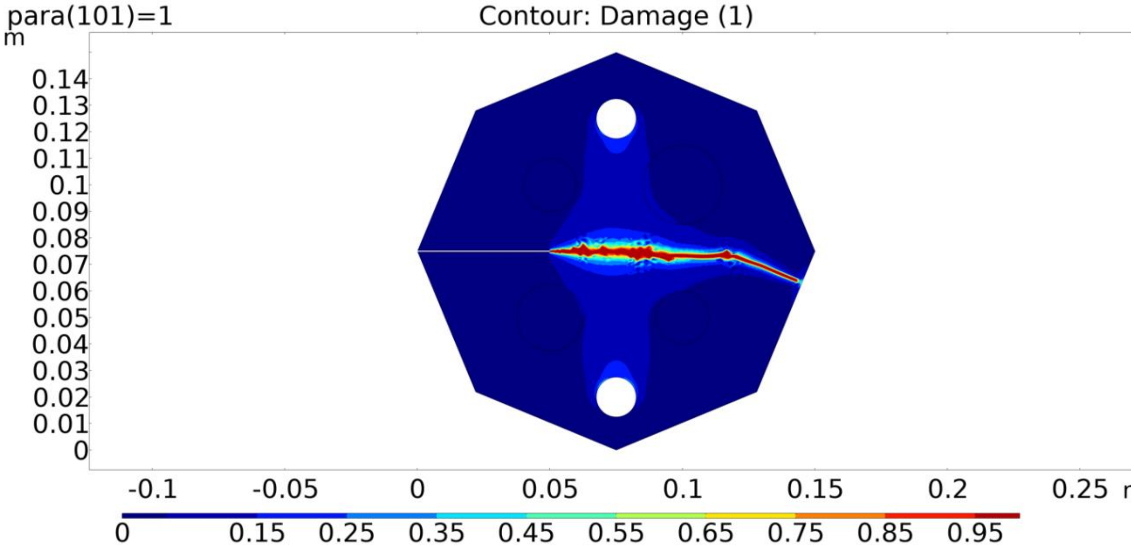


Fig. 11. Crack propagation in conglomerates due to the initial crack definition

6. Conclusion

The purpose of this study was to demonstrate that the study of micromechanics properties of rockfill is very important, especially when it comes to designing dams with high rockfill. The MES dam was selected for this purpose, for which no micromechanics study was conducted during the design process, only macro mechanics laboratory tests were conducted. In this paper, sets of XRD analyses were performed to investigate the microstructure of conglomerates rockfill, which revealed that calcite constitutes the majority of the rockfill structure. In accordance with geology science, the calcite has a high potential for breaking, and a numerical simulation was developed to illustrate the fractures and crack propagation in a high rockfill dam. This crack propagation in conglomerates caused the fractures and failure of the conglomerates rockfill material. The effect of this crack propagation in conglomerates on the MES dam settlement was determined through the relationship of plastic work in the crack tip and plastic work in the laboratory test. Results of this paper revealed that occurring crest settlement of MES rockfill dam just by particle breakage has potential to a settlement near an 11.5m. The current settlement of the crest is 6.5 m. This important result revealed that the dam deformation would continue and proper measures should take to prevent dam breakage.

References

1. M. Zhou and E. Song, "A random virtual crack DEM model for creep behavior of rockfill based on the subcritical crack propagation theory," *Acta Geotech.*, vol. 11, no. 4, pp. 827–847, Aug. 2016, doi: 10.1007/s11440-016-0446-8.
2. Z. Huda, "Creep Behavior of Materials," 2022, pp. 253–265.
3. D. Liu, J. Yang, and B. Dong, "Discrete element analysis of the influence of compaction quality on mechanical properties of rockfill materials," *Comput. Geotech.*, vol. 151, p. 104958, Nov. 2022, doi: 10.1016/J.COMPGEO.2022.104958.
4. J. Hong and M. Xu, "Numerical investigation of the time size effect of high rockfill geostructures," *Transp. Geotech.*, vol. 30, p. 100613, Sep. 2021, doi: 10.1016/j.trgeo.2021.100613.
5. A. Khonji, A. Bagherzadeh-Khalkhali, and A. Aghaei-Araei, "Experimental investigation of rockfill particle breakage under large-scale triaxial tests using five different breakage factors," *Powder Technol.*, vol. 363, pp. 473–487, Mar. 2020, doi: 10.1016/J.POWTEC.2020.01.032.
6. B. S. derivand Ahmadreza Mazaheri , Rasoul Alipour, "Study the Monitoring and Numerical Analyses of Rockfill Dam; Case Study of Marvak Dam in Lorestan, Iran," *Civ. Infrastruct. Res.*, vol. 5, no. 2, pp. 153–164, 2020, doi: 10.22091/CER.2020.5223.1194.
7. Y. Jia, B. Xu, S. Chi, B. Xiang, and Y. Zhou, "Research on the Particle Breakage of Rockfill Materials during Triaxial Tests," *Int. J. Geomech.*, vol. 17, no. 10, Oct. 2017, doi: 10.1061/(ASCE)GM.1943-5622.0000977.
8. Y. Jia, B. Xu, C. S. Desai, S. Chi, and B. Xiang, "Rockfill Particle Breakage Generated by Wetting Deformation under the Complex Stress Path," *Int. J. Geomech.*, vol. 20, no. 10, Oct. 2020, doi: 10.1061/(ASCE)GM.1943-5622.0001789.
9. G. R. McDowell and M. D. Bolton, "On the micromechanics of crushable aggregates," *Géotechnique*, vol. 48, no. 5, pp. 667–679, Oct. 1998, doi: 10.1680/geot.1998.48.5.667.
10. L. A. Oldecop and E. E. Alonso, "A model for rockfill compressibility," *Géotechnique*, vol. 51, no. 2, pp. 127–139, Mar. 2001, doi: 10.1680/geot.2001.51.2.127.
11. H. Rahmani and A. K. Panah, "Influence of particle size on particle breakage and shear strength of weak rockfill," *Bull. Eng. Geol. Environ.*, vol. 80, no. 1, pp. 473–489, Jan. 2021, doi: 10.1007/s10064-020-01889-6.

12. W. Salim and B. Indraratna, “A new elastoplastic constitutive model for coarse granular aggregates incorporating particle breakage,” *Can. Geotech. J.*, vol. 41, no. 4, pp. 657–671, Aug. 2004, doi: 10.1139/t04-025.
13. Q. D. SUN, B. INDRARATNA, and S. NIMBALKAR, “Effect of cyclic loading frequency on the permanent deformation and degradation of railway ballast,” *Géotechnique*, vol. 64, no. 9, pp. 746–751, Jul. 2014, doi: 10.1680/geot.14.T.015.
14. B. Y. Zhang, J. H. Zhang, and G. L. Sun, “Deformation and shear strength of rockfill materials composed of soft siltstones subjected to stress, cyclical drying/wetting and temperature variations,” *Eng. Geol.*, vol. 190, pp. 87–97, May 2015, doi: 10.1016/j.enggeo.2015.03.006.
15. A. K. Gupta, “Effects of particle size and confining pressure on breakage factor of rockfill materials using medium triaxial test,” *J. Rock Mech. Geotech. Eng.*, vol. 8, no. 3, pp. 378–388, Jun. 2016, doi: 10.1016/j.jrmge.2015.12.005.
16. Y. Xiao, H. Liu, Y. Chen, and J. Jiang, “Strength and Deformation of Rockfill Material Based on Large-Scale Triaxial Compression Tests. I: Influences of Density and Pressure,” *J. Geotech. Geoenvironmental Eng.*, vol. 140, no. 12, Dec. 2014, doi: 10.1061/(ASCE)GT.1943-5606.0001176.
17. Y. Xiao, H. Liu, C. S. Desai, Y. Sun, and H. Liu, “Effect of Intermediate Principal-Stress Ratio on Particle Breakage of Rockfill Material,” *J. Geotech. Geoenvironmental Eng.*, vol. 142, no. 4, Apr. 2016, doi: 10.1061/(ASCE)GT.1943-5606.0001433.
18. B. O. Hardin, “Crushing of Soil Particles,” *J. Geotech. Eng.*, vol. 111, no. 10, pp. 1177–1192, Oct. 1985, doi: 10.1061/(ASCE)0733-9410(1985)111:10(1177).
19. B. Indraratna, N. T. Ngo, and C. Rujikiatkamjorn, “Deformation of Coal Fouled Ballast Stabilized with Geogrid under Cyclic Load,” *J. Geotech. Geoenvironmental Eng.*, vol. 139, no. 8, pp. 1275–1289, Aug. 2013, doi: 10.1061/(ASCE)GT.1943-5606.0000864.
20. H. Liu, K. Zeng, and Y. Zou, “Particle Breakage of Calcareous Sand and Its Correlation with Input Energy,” *Int. J. Geomech.*, vol. 20, no. 2, Feb. 2020, doi: 10.1061/(ASCE)GM.1943-5622.0001541.
21. N. D. Marschi, C. K. Chan, and H. B. Seed, “Evaluation of Properties of Rockfill Materials,” *J. Soil Mech. Found. Div.*, vol. 98, no. 1, pp. 95–114, Jan. 1972, doi: 10.1061/JSFEAQ.0001735.
22. B. Y. Zhang, J. H. Zhang, and G. L. Sun, “Particle breakage of argillaceous siltstone subjected to stresses and weathering,” *Eng. Geol.*, vol. 137–138, pp. 21–28, Jun. 2012, doi: 10.1016/j.enggeo.2012.03.009.
23. S. Alidadi, R. Alipour, and M. Shakeri, “Influence of rockfill particle breakage on long-term settlement of embankment dams,” *Proc. Inst. Civ. Eng. - Geotech. Eng.*, pp. 1–11, Feb. 2022, doi: 10.1680/jgeen.21.00005.
24. M. B. and T. A. N. Hafiz M.N. Iqbal, *Nano-Bioremediation : Fundamentals and Applications*. Elsevier, 2022.
25. M. Ambati, T. Gerasimov, and L. De Lorenzis, “A review on phase-field models of brittle fracture and a new fast hybrid formulation,” *Comput. Mech.*, vol. 55, no. 2, pp. 383–405, Feb. 2015, doi: 10.1007/s00466-014-1109-y.

26. H. Heidarzadeh and R. Kamgar, “Necessity of applying the concept of the steady state on the numerical analyses of excavation issues: laboratory, field and numerical investigations,” *Geomech. Geoengin.*, vol. 17, no. 2, pp. 413–425, Mar. 2022, doi: 10.1080/17486025.2020.1755466.
27. H. Heidarzadeh, “Evaluation of modified Cam-Clay constitutive model in FLAC and its development by FISH programming,” *Eur. J. Environ. Civ. Eng.*, vol. 25, no. 2, pp. 226–244, Jan. 2021, doi: 10.1080/19648189.2018.1521752.
28. Naderi R, Khademalrasoul A. Fully Automatic Crack Propagation Modeling in the Interaction with Void and Inclusion without Remeshing. Modares Mechanical Engineering. 2015; 15 (7) :261-273
29. C. Miehe, M. Hofacker, and F. Welschinger, “A phase field model for rate-independent crack propagation: Robust algorithmic implementation based on operator splits,” *Comput. Methods Appl. Mech. Eng.*, vol. 199, no. 45–48, pp. 2765–2778, Nov. 2010, doi: 10.1016/j.cma.2010.04.011.